

# Is transport in time-dependent random potentials universal ?

Yevgeny Krivolapov and Shmuel Fishman

*Physics Department, Technion - Israel Institute of Technology, Haifa 32000, Israel.*

The growth of the average kinetic energy of classical particles is studied for potentials that are random both in space and time. Such potentials are relevant for recent experiments in optics and in atom optics. It is found that for small velocities uniform acceleration takes place, and at a later stage fluctuations of the potential are encountered, resulting in a regime of anomalous diffusion. This regime was studied in the framework of the Fokker-Planck approximation. The diffusion coefficient in velocity was expressed in terms of the average power spectral density, which is the Fourier transform of the potential correlation function. This enabled to establish a scaling form for the Fokker-Planck equation and to compute the large and small velocity limits of the diffusion coefficient. A classification of the random potentials into universality classes, characterized by the form of the diffusion coefficient in the limit of large and small velocity, was performed. It was shown that one dimensional systems exhibit a large variety of novel universality classes, contrary to systems in higher dimensions, where only one universality class is possible. The relation to Chirikov resonances, that are central in the theory of Chaos, was demonstrated. The general theory was applied and numerically tested for specific physically relevant examples.

Dynamics in potentials which are random both in space and time were subject of many sophisticated studies for nearly 100 years [1–4]. The response to forces resulting of such potentials typically differs from ordinary diffusion. Specifically, the diffusion coefficients predicted by such mechanisms sensitively depend on the velocity of the particles. Also, if the potential is time-dependent, the energies of the particles will not be constant. The existence of such ‘anomalous diffusion’ has been demonstrated for classical dynamics with spatially and temporally fluctuating potentials [5–10]. These works claim universal behavior in the sense that for generic random potentials the diffusion coefficient exhibits a universal power-law dependence on instantaneous velocity  $v$ , such that  $D(v) \sim |v|^{-3}$  as  $|v| \rightarrow \infty$ . This in turn implies that asymptotically in time the average velocity satisfies  $\langle v^2 \rangle \sim t^{2/5}$ . In the present Letter we will demonstrate that this picture should be revised, and we will introduce such a revision. The potentials we study here are continuous in space and are fundamentally different from lattice models where the velocity (momentum) is inherently bounded.

In the present work we consider the classical dynamics of a particle in potentials that are random both in space and in time, emphasizing the spreading of the velocity acquired by the particle, as time evolves. This is a fundamental problem, which was motivated by experiments in optics [11] and in atom optics [12, 13], where the random potential is introduced by transforming an intensity pattern into an effective potential, for the light [14, 15] or the cold atoms [12, 13]. Such potentials are naturally described in terms of the Fourier spectrum of the waves inducing them, and their spectral coefficients are assumed to be independent random variables. In addition to the fundamental interest, the calculations of the present work are also relevant for spreading of waves in random time-dependent potentials, because at least

in the regime of large velocities (short wave-length) it is generally believed that a classical picture is appropriate. Therefore, we may expect that the classical results, which are presented in this Letter are relevant also for the wave experiments (e.g., [11–13]). We will consider a general stationary (both in space and time) random potential, which is also isotropic on average. Such potentials are conveniently described by their Fourier components,

$$V^{(1)}(\mathbf{x}, t) = \int \hat{V}(\mathbf{k}, \omega) \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t) d\mathbf{k} d\omega + c.c. \quad (1)$$

where  $\hat{V}(\mathbf{k}, \omega)$  is a random field chosen such that the distribution of  $V^{(1)}(\mathbf{x}, t)$  is stationary both in time and space, in such case  $\langle \hat{V}(\mathbf{k}, \omega) \hat{V}(\mathbf{k}', \omega') \rangle = V_0^2 \delta(\mathbf{k} - \mathbf{k}') \delta(\omega - \omega')$ , where  $\langle . \rangle$  denotes the ensemble average. For time dependent potentials the velocity of the particles may grow, since energy is not conserved. The dominant mechanism of the growth of the velocity is via Chirikov resonances [16], namely, resonances between the particle dynamics and the external driving. Those resonances occur when the phases in (1) are stationary. For a given velocity  $\mathbf{v} = \dot{\mathbf{x}}$ , this happens for

$$\mathbf{k} \cdot \mathbf{v} = \omega. \quad (2)$$

In one dimensional systems this condition reduces to  $kv = \omega$ , namely, when the particle just ‘surfs’ on one of the waves composing (1). The growth of the velocity occurs only when the particle ‘jumps’ to a wave traveling with a nearby and larger velocity than the velocity of its initial carrier. This provides an intuitive picture for the understanding of transport in phase-space, since the growth of the velocity is limited to regions in phase-space where the density of Chirikov resonances is non-zero. In dimensions two and higher particles may increase their velocity beyond the velocity of the wave by traveling in a direction that is not parallel or even perpendicular to

the wave propagation direction. This essential difference will yield distinct much richer transport behavior in one dimensional systems than in higher dimensions, as will be explained in what follows [17].

Since, in this work we consider only potentials continuous in both time and space it is natural to define the characteristic length and time scales of the potential,  $l_x$  and  $l_t$ , correspondingly. Those, scales are defined such that the variation of the potential over them is limited, and the particle will encounter an almost constant force. Note, that  $l_t$  and  $l_x$  are generally different from the exponential decay rates of the correlation function of the potential. If the force is weak in the sense,  $F < l_x/l_t^2$ , particles with velocities  $v < l_x/l_t$  (as will be assumed in the present work) will experience an uniform acceleration up to time  $t < l_t$ . For longer time-scales the variations of the potential become apparent for the particle, and the velocity will exhibit anomalous diffusion.

To calculate the anomalous diffusion coefficient we invoke the Fokker-Planck approximation, assuming that the force is sufficiently weak and the decay of the potential correlations,  $C(\mathbf{x}_1 - \mathbf{x}_2, t_1 - t_2) = \langle V(\mathbf{x}_1, t_1) V(\mathbf{x}_2, t_2) \rangle$ , is sufficiently rapid, so that the velocity can be considered constant on the time scale where the correlation function is appreciable. For stationary random potentials it is convenient to use the average power spectral density (PSD),  $S(\mathbf{k}, \omega)$ , defined as the Fourier transform of the correlation function (Weiner-Khinchin theorem),

$$C(\mathbf{x}, t) = \int d\omega \int d\mathbf{k} S(\mathbf{k}, \omega) \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t). \quad (3)$$

The benefit of this representation is twofold. First, in experiments [11–13, 18, 19] the PSD is the naturally controlled rather than the correlation function. Second, the analytical relations obtained in this representation are more transparent. For a potential which is not only stationary, but also has an isotropic PSD, and initial distribution which is isotropic as well, the Fokker-Planck equation for the velocity is effectively one dimensional [4],

$$\frac{\partial P}{\partial t} = \left( v^{-(d-1)} \frac{\partial}{\partial v} v^{d-1} D(v) \frac{\partial}{\partial v} \right) P, \quad (4)$$

where  $P(\mathbf{v}, t)$  is the probability density and  $D(v)$  is the diffusion coefficient given by

$$D(v) = \pi \int d\mathbf{k} (\mathbf{k} \cdot \hat{v})^2 S(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}). \quad (5)$$

Note, that if the PSD has some typical scales,  $k_0$  and  $\omega_0$  then the Fokker-Planck equation is invariant under the transformation of variables,

$$\begin{aligned} \mathbf{v} &\rightarrow \mathbf{v}' \\ t &\rightarrow t' \left( \pi V_0^2 \frac{k_0^2}{\omega_0} \right)^{-1}, \end{aligned} \quad (6)$$

which allows the rescaling of the Fokker-Planck equation to a universal form (where  $V_0$  is the amplitude of the potential). For dimensions two and higher, changing variables to  $y = v \cos \theta$  and expanding the angular part of the the integrand in (5) gives the asymptotic behavior,

$$D(v) \sim \frac{D_3}{v^3}, \quad (7)$$

with

$$D_3 = 2S_d \int_0^\infty dy \int_0^\infty dk y^2 k^{d+1} S(k, ky), \quad (8)$$

where  $S_d$  is the surface of a  $d$ -dimensional hyper-sphere. Using this asymptotics an asymptotic scaling solution to the Fokker-Planck equation may be obtained,  $P(v, t) = t^{-d/5} g(v^5/t)$ , which yields the growth of the mean kinetic energy as,  $\frac{1}{2} \langle v^2 \rangle \sim t^{2/5}$ . This behavior is considered in the literature as universal [5, 7, 8, 10] for any dimension given that the correlation function of the potential is sufficiently differentiable. For dimensions two and higher as shown above this is indeed the case, however for one dimensional systems other behaviors are possible.

For one dimensional systems new possibilities arise from the fact that the angular part in (5) is missing and therefore the diffusion coefficient,

$$D(v) = \pi \int k^2 S(k, kv) dk. \quad (9)$$

may acquire various asymptotic behaviors dictated by  $S(k, \omega)$ . Consider a diffusion coefficient which decreases with velocity faster than any power law, then clearly the asymptotic expansion in  $v^{-1}$  is not useful, since it will produce a nil result. A simple example is given by a PSD with the property,  $S(k, \omega) = 0 \quad \omega/k > v_{\max}$ , which renders the diffusion coefficient zero for large velocities,  $D(v) = 0 \quad v > v_{\max}$ , as demonstrated in [17] for a specific example. We will now demonstrate this behavior, using an experimentally relevant potential which is proportional to the intensity,  $V(x, t) = |U(x, t)|^2$  of some complex field, which in turn is a superposition of waves  $U(x, t) = \int dk \hat{U}(k) \exp i(kx - \omega(k)t)$ , with some dispersion relation  $\omega(k)$ . These potentials appear in experiments with neutral atoms [12, 13] and in some experiments in optics [11, 18, 19], where the variation of the refractive index (which plays the role of the potential) of a photosensitive material is proportional to the intensity of light [14, 15]. For simplicity we will assume the dispersion relation to be  $\omega(k) = k^2/2$  (which naturally appears in the framework of the paraxial approximation in optics), and  $f(k)$  to be the probability density of the wave numbers  $k$ . For this type of potentials one can readily obtain both the PSD,  $S(k, \omega) = V_0^2 |k|^{-1} f(\frac{k}{2} - \frac{\omega}{k}) f(\frac{k}{2} + \frac{\omega}{k})$ , which is singular for  $k = 0$ , and the diffusion coefficient,  $D(v) = 4\pi V_0^2 \int |q| f(q+v) f(q-v) dq$ . It is clear that the decay

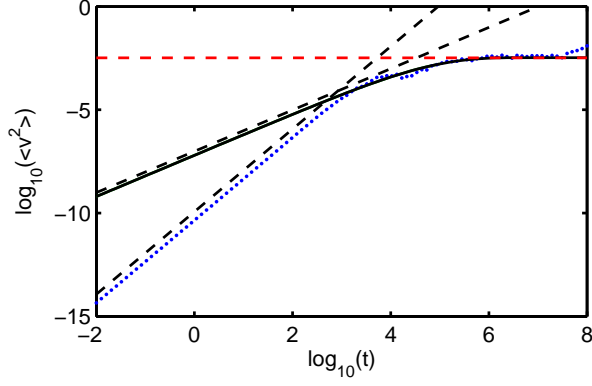


Figure 1: A log-log plot of average squared velocity as a function of time for one dimensional system with a potential  $V(x, t) = |U(x, t)|^2$  and a uniform distribution of wave-numbers. The blue dots represent the result of Monte-Carlo simulation the corresponding Newton equations averaged over 20 realizations and the black solid line is the numerical solution of the Fokker-Planck equation for the velocity (4). The dashed black and red lines are guides for the eye with the corresponding slopes of 2, 1 and 0. The initial condition was a narrow distribution of velocities for the Fokker-Planck Eq. and  $x = v = 0$  for the Monte-Carlo calculation. The parameters used for this simulation are,  $V_0 = 10^{-4}$ ,  $k_R = 0.1$ .

of the diffusion coefficient with the velocity is dictated by the decay of the probability density of the wave-numbers,  $f(k)$ . An explicit expression may be obtained for example for a uniform distribution of wave-numbers (in the interval  $[-k_R, k_R]$ ) giving [17],

$$D(v) = \begin{cases} \frac{\pi V_0^2}{k_R^2} (k_R - |v|)^2 & |v| < k_R \\ 0 & |v| > k_R \end{cases}. \quad (10)$$

The resulting dynamics is demonstrated in Fig. 1. The regime of a unit slope corresponds to regular diffusion and the asymptotic regime corresponds to an absence of diffusion in velocity.

Another explicit expression may be obtained for a Gaussian distribution of wave-numbers, giving a diffusion coefficient

$$D(v) = 2V_0^2 \exp(-v^2/k_R^2). \quad (11)$$

The predictions of this equation are compared with Monte-Carlo simulations and the results are presented in Fig. 2. The correlation function for this potential can be also obtained using (3),

$$C(x, t) = V_0^2 \frac{1}{\sqrt{1 + k_R^4 t^2}} \exp\left(-\frac{k_R^2 x^2}{1 + k_R^4 t^2}\right). \quad (12)$$

Note, that the correlation function (12) is infinitely differentiable, decaying fast in position and slowly decaying in time.

These examples demonstrate asymptotic behavior which is very different from (7) described in previous

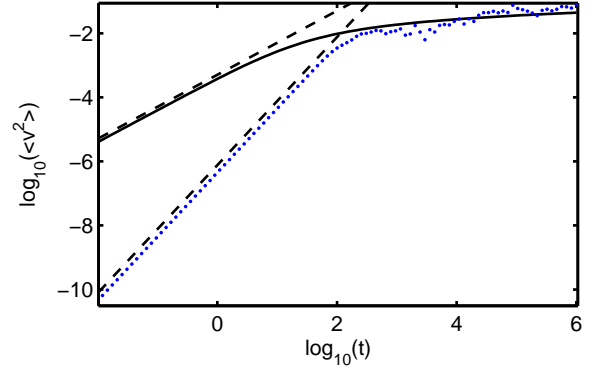


Figure 2: Same as Fig. 1 but with a Gaussian distribution of wave-numbers. The dashed black lines are guides for the eye with the corresponding slopes of 2 and 1. The parameters used for this simulation are,  $V_0 = 10^{-2}$ ,  $k_R = 0.1$ .

studies [5, 8, 10]. Therefore new universality classes of potentials should be defined. These universality classes are characterized by a growth in velocity which is subdiffusive such that the moments of the velocity distribution grow slower than any power law. It is important to note that these new universality classes are neither limited to potentials with a dispersion relation nor to correlations that are slowly decaying. For example, the PSD,

$$S(k, \omega) \sim \exp(-k_0/k)^2 \exp(-\omega^2/\omega_0^2 - k^2/k_0^2), \quad (13)$$

gives rise to a rapidly decaying correlation function and a diffusion coefficient which is  $D(v) \sim g(v) \exp\left(-\sqrt{1 + (v/v_0)^2}\right)$ , where  $g(v)$  is some polynomial of  $v$ . The classification of potentials into different universality classes, which differ from (7), may be obtained for a PSD which is not singular and differentiable on the line  $k = 0$ . This turns out very useful since in some experiments the PSD could be precisely controlled, which suggest a way to test the various claims of this Letter. The diffusion coefficient can be expanded in powers of  $v^{-1}$ ,

$$D(v) = \frac{D_3}{v^3} + \frac{D_5}{v^5} + \dots, \quad (14)$$

where

$$D_n = \frac{1}{(n-3)!} \int d\omega \omega^{n-1} \frac{\partial^{n-3} S}{\partial k^{n-3}}(k, \omega)|_{k=0} \quad n \geq 3. \quad (15)$$

If the first  $n_{\max}$  derivatives,  $\partial^n S / \partial k^n$ , vanish on the line  $k = 0$ , where  $n_{\max}$  is the first derivative that is different from zero, then the resulting asymptotic behavior is  $D(v) \sim D_{n_{\max}+3}/v^{n_{\max}+3}$ . The case where all the derivatives vanish is possible only if  $S(k, \omega)$  is non-analytic on the line  $k = 0$ . The non-analytic behavior

may be found either for a function, which is strictly zero on some finite strip around the line  $k = 0$ , or due to an essential singularity of  $S(k, \omega)$  on this line. The first case will lead to a diffusion coefficient that will vanish for large velocities, for example (10) (where the support of  $f(k)$  is finite). The second case results in a decay faster than any power law (11). By controlling the analytical behavior of the PSD on the line  $k = 0$ , one can vary the asymptotic behavior of the velocity dependence of  $D(v)$ . The range of variation is from  $D(v) \sim v^{-3}$ , through  $D(v) \sim v^{-n}$  (with  $n > 3$ ) to sub-exponential, exponential, super-exponential and up-to  $D(v) = 0$  (for  $v > v_{\max}$ ).

In this Letter the diffusion coefficient for the velocity is presented in terms of the average power spectral density (PSD) (see Eq. (5)) for stationary potentials that are random both in space and time. This representation is very natural for stationary (both in time and space) potentials, in particular, for potentials which are a superposition of waves, as they appear in optics and atom optics. The simplicity of the expression enabled to explore the properties of the diffusion coefficient, to establish a scaling form of the Fokker-Planck equation, and to discover new universality classes. In particular, we were able to calculate explicitly the diffusion coefficient for representative examples, relevant for applications both in optics and atom optics (Gaussian and uniform distributions of wave-vectors). We have shown that for dimensions larger than one only one universality class is possible in the framework of the Fokker-Planck approximation,  $D(v) \sim v^{-3}$ . However for one dimensional systems new possibilities for large velocity asymptotics were also found. It was demonstrated that diffusion in phase-space takes place only where Chirikov resonances (2) are found.

The main result of this Letter is the classification of universality classes in one dimensional systems. In the past it was found that in the large velocity limit the diffusion coefficient depends on the velocity as,  $D(v) \sim v^{-3}$  [5, 6, 8, 10]. In the present work we have shown that this is always the case for dimensions larger than one, and explained the mechanism of this behavior (see discussion before (7)). However, for one dimensional systems this is only one of the possibilities. Generally, the possible asymptotic expansion of  $D(v)$  in powers of  $v^{-1}$  is:

1. The first term of the asymptotic expansion of  $D(v)$  is non-zero,  $D(v) \sim v^{-3}$ .
2. The first non-vanishing term in the asymptotic expansion of  $D(v)$  is  $n$  then,  $D(v) \sim v^{-n}$ .
3. All terms in the asymptotic expansion of  $D(v)$  are zero,  $D(v) \leq v^{-\alpha}$ , for any  $\alpha > 0$ . In particular, the diffusion coefficient may be zero for  $v > v_{\max}$  (e.g. (10)) or non-zero but decreasing faster than any power law (e.g., (11)).

All these possibilities can be realized in experiments with good control over the PSD or more precisely the probability density of the wave-numbers, which comprise the potential,  $f(k)$ . Unlike statements of other studies [5, 6, 8, 10], the differentiability of the correlation function of the potential is not related to the classification into universality classes. Additionally, it was demonstrated that the new universality classes do not depend on the range of the correlation function.

For small velocities the diffusion coefficient, obtained in the framework of the Fokker-Planck approximation, will be generally different from zero and therefore a regime of regular diffusion is expected. However, we have shown that initially particles will experience a uniform acceleration, and therefore at least for short times the Fokker-Planck approximation is invalid, as could be clearly seen from all the figures. We have also found potentials, which produce diffusion coefficients *growing* with velocity (from zero and up to some value) [17] for which the validity of the Fokker-Planck approximation is not satisfied, since the condition that the velocity is constant during the correlation time is violated. On a longer time-scale, the velocity will follow a diffusion equation and eventually, it will reach the asymptotic long-time behavior of anomalous diffusion as predicted by the Fokker-Planck approximation.

In the present work the spreading of the velocity distribution is studied in the framework of the Fokker-Planck equation, therefore an obvious question to study is what happens when the Fokker-Planck approximation fails. Another important issue, which was not addressed in this work, is how to obtain analytically the spreading in position. Since in the experiments that have motivated this work [11, 19], the relevant dynamics is of waves, rather than particles, an obvious question to explore is the correspondence between the classical and wave dynamics. This correspondence is shown in Fig. 3, where in addition to Monte-Carlo and Fokker-Planck calculations of the average squared velocity for particles a numerical simulation of a corresponding wave system is presented. As is expected there is a reasonable correspondence for large velocities. It will be explored in further studies and the possibility for its violation is of great interest.

This work was motivated by the experimental work of Liad Levi and Mordechai Segev, whom we thank for many stimulating discussions and for providing crucial insight for this problem. It is our great pleasure to thank Tom Spencer for introducing us to [5, 6] and to Michael Wilkinson for introducing us to [8, 10]. Many of the results of the present work originated from fruitful discussions with Michael Wilkinson during his visit to the Technion. The work was supported in part by the US-Israel Binational Science Foundation (BSF).

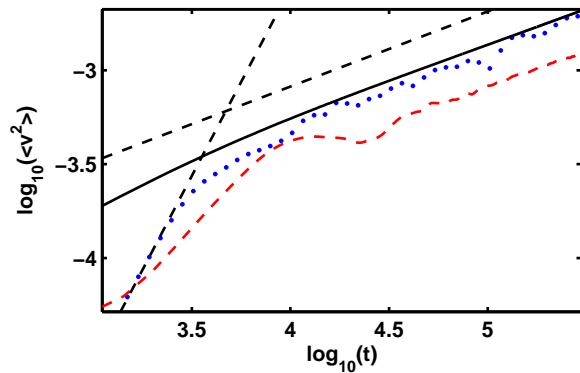


Figure 3: Same as Fig. 2 but for a two dimensional system, and including the wave dynamics. The dashed black lines are guides for the eye with the corresponding slopes of 2 and  $2/5$ . The dashed (light) red line is the result of a numerical simulation of the Schrödinger equation with the same potential, averaged over 100 realizations. The initial distributions of particle velocities are the same as the initial wavefunction, a narrow Gaussian around the origin. The parameters used for this simulation are,  $V_0 = 2.06 \times 10^{-4}$ ,  $k_R = 0.029$  (same as experimental parameters in [19]).

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- [1] P. Langevin, C. R. Acad. Sci. (Paris) **146**, 530 (1908).
  - [2] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).
  - [3] P. A. Sturrock, Physical Review **141**, 186 (1966).
  - [4] N. V. Kampen, *Stochastic Processes in Physics and Chemistry, Third Edition* (North Holland, 2007), 3rd ed.

- [5] L. Golubović, S. Feng, and F.-A. Zeng, Phys. Rev. Lett. **67**, 2115 (1991).
- [6] M. N. Rosenbluth, Phys. Rev. Lett. **69**, 1831 (1992).
- [7] E. Arvedson, M. Wilkinson, B. Mehlig, and K. Nakamura, Phys. Rev. Lett. **96**, 030601 (2006).
- [8] V. Bezuglyy, B. Mehlig, M. Wilkinson, K. Nakamura, and E. Arvedson, J. Math. Phys. **47**, 073301 (2006).
- [9] B. Aguer, S. Bièvre, P. Lafitte, and P. E. Parris, J. Stat. Phys. **138**, 780 (2009).
- [10] V. Bezuglyy, M. Wilkinson, and B. Mehlig, *Universal anomalous diffusion of weakly damped particles*, arXiv:1203.1354v1 (2012).
- [11] T. Schwartz, G. Bartal, S. Fishman, and M. Segev, Nature **446**, 52 (2007).
- [12] J. E. Lye, L. Fallani, M. Modugno, D. S. Wiersma, C. Fort, and M. Inguscio, Phys. Rev. Lett. **95**, 070401 (2005).
- [13] L. Sanchez-Palencia, D. Clement, P. Lugan, P. Bouyer, G. V. Shlyapnikov, and A. Aspect, Phys. Rev. Lett. **98**, 210401 (2007).
- [14] N. K. Efremidis, S. Sears, D. N. Christodoulides, J. W. Fleischer, and M. Segev, Phys. Rev. E **66**, 046602 (2002).
- [15] J. W. Fleischer, T. Carmon, M. Segev, N. K. Efremidis, and D. N. Christodoulides, Phys. Rev. Lett. **90**, 023902 (2003).
- [16] G. M. Zaslavsky and B. V. Chirikov, Sov. Phys. Usp. **14**, 549 (1972).
- [17] Y. Krivolapov, L. Levi, S. Fishman, M. Segev, and M. Wilkinson, *Super-diffusion in optical realizations of Anderson localization*, arXiv:1203.0838 (2012), to be published in New. J. Phys.
- [18] L. Levi, M. Rechtsman, B. Freedman, T. Schwartz, O. Manela, and M. Segev, Science **332**, 1541 (2011).
- [19] L. Levi, Y. Krivolapov, M. Wilkinson, S. Fishman, , and M. Segev (2012), to be submitted to Nature.